Stochastic ordering properties of the exponential, upper truncated and lower truncated families of distributions

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Abstract

Using Bayes approach we obtain the predictive density functions for the exponential, range of type A and range of type B families of distributions, conditioned on a sufficient statistics. We then examine stochastic ordering properties useful to inventory control theory.

Key words: Sufficient statistic, predictive densities, stochastic ordering, Bayesian inventory control.

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1. INTRODUCTION

The concept of stochastic ordering on probability densities is rather old. During the years, various types of stochastic ordering have been proposed. Each one of them has served different needs and scientific settings. The definition given here is based on the cumulative distribution function.

Definition. *Let* φ *and* ψ *be two density functions and* $\Phi(x)$, $\Psi(x)$ *their corresponding cumulative distribution functions (c.d.f). If* $\Phi(x)\geq\Psi(x)$ *for all* $x\in R$, *we say that the density* φ *is stochastically smaller than* ψ *and we write it as* $\varphi\subset\psi$.

The first application of this type of stochastic ordering seems to appear in the paper of Mann and Withney (1947), who tried to characterize the alternatives when testing for equality of two distributions.

Stochastic ordering has proved an important research tool in many scientific areas: Queuing theory (Stoyan, 1983), reliability theory (Boland et al. 1992), stochastic scheduling (Ross, 1983), economics and operations research (Savage, 1972, Masse, 1962). A most extensive treatment of this concept and its applications is given in the book by Shaked and Shanthikumar (1994).

In the area of inventory control the concept of stochastic ordering enters through the demand function for the product in question. Although not absolutely clear, it seems that it has appeared for the first time, in the pioneering paper by Scarf (1959). Scarf, using adaptive dynamic programming, managed to compare optimal ordering levels under different statistical information settings. This work was followed by the work of Karlin (1960). Karlin used the stochastic ordering in a more systematic way. In the first part of his paper he produced some very interesting and fairly general results. In the second part he specialized on the densities describing the system's demand by considering two classes of probability densities, namely the exponential and range family of type A. For each of these families he studied the stochastic ordering properties of their predictive densities and then he applied the obtained results to produce some very interesting ordering relations for the optimal ordering levels of the inventory system under study. Following the work of Scarf and Karlin, Iglehart (1964) extended their results, and produced new ones. His contribution was important in two directions: (i) Establishing stochastic ordering properties for the predictive density functions and (ii) transferring this ordering to monotonicity properties on the optimal ordering levels. Closing this important paper, Iglehart proposed the study of the range family of type B, another family of truncated densities, which could be used to describe the demand fluctuations for some inventory systems. Iglehart suspected that this family might also have nice stochastic ordering properties. Work in the field continued with Brown and Rogers (1972), Papachristos (1977a, 1977b), Azoury (1985).

The common characteristic of the research mentioned in inventory control theory, was that the densities used to describe demand fluctuations accepted a sufficient statistic for the unknown parameter ω. This statistics was based on all past demand records supposed available. However there are cases, where not all past demand records are completely available. Such are the cases with censored, or missing data. In such cases we need to consider families, which can accept, sufficient statistics based on censored or missing data. In this direction we mention the most important contributions, which are the work of Braden and Freimer (1991) and Nahmias (1994).

In this paper we study the stochastic ordering properties of the predictive densities of three families of distributions i.e. the exponential, the range family of type A, and the range of type B. Section 2 is devoted to the exponential family. We review the existing results and give a new one. We then proceed with section 3 where we deal with the range family of type A. We also review the existing results, and we present a new one. Section 4 is devoted to the study the range family of type B, proposed by Iglehart (1964). For this family we establish stochastic ordering properties, for its predictive density function. This ordering is related to the information available for the parameter ω. Finally in section 5 we give some concluding remarks and we are making proposals, for possible applications of the produced results and further research in the area of inventory control.

2. THE EXPONENTIAL FAMILY

Let ξ be a continuous random variable with density belonging to the exponential family

$$\varphi(\xi|\omega)=\beta(\omega)\exp(-\omega\xi)r(\xi) \quad \xi\geq 0 \tag{1}$$

where

$$(\beta(\omega))^{-1}=\int_0^\infty \exp(-\omega\xi)r(\xi)d\xi \tag{2}$$

ω is unknown parameter taking values at the interval $I_1=[0,\infty)$ and $r(\xi)$ is a strictly positive known function such that the integral in (2) exists for every $\omega \in I_1$. It 's obvious that $\beta(\omega)$ is a strictly increasing function of ω. From this family we can obtain the distributions, Gamma in the continuous case, Poisson and Negative Binomial in the discrete case.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be n independent observations of the r.v. ξ and $Q=\sum_{i=1}^{n}\xi_i$ and $s=\frac{Q}{n}$. Then

$$\varphi(\xi_1, \xi_2, \ldots, \xi_n \mid \omega)=\{\beta(\omega)\}^n \exp(-\omega ns)\prod_{i=1}^{n} r(\xi_i) \qquad (3)$$

From Fisher-Neyman factorization theorem (Raiffa and Schlaifer, 1961) we have that the pair (Q, n) or equivalently (s, n) is a sufficient statistic for the parameter ω. If $f(\omega)$ is the prior probability density (p.d.) of ω then using the Bayes theorem we find the posterior density of ω given Q, n as

$$f(\omega \mid Q, n)=\frac{\{\beta(\omega)\}^n \exp(-\omega Q) f(\omega) d\omega}{\int_0^\infty \{\beta(\omega)\}^n \exp(-\omega Q) f(\omega) d\omega} \qquad (4)$$

The predictive probability density function (p.p.d.f.) of ξ is now given by

$$\varphi(\xi \mid Q, n)=\int_0^\infty \varphi(\xi \mid \omega) f(\omega \mid Q,n) d\omega = r(\xi)\frac{\int_0^\infty \{\beta(\omega)\}^{n+1} \exp(-\omega\xi)\exp(-\omega Q) f(\omega) d\omega}{\int_0^\infty \{\beta(\omega)\}^n \exp(-\omega Q) f(\omega) d\omega}$$

$$= r(\xi)\frac{A(n+1, Q+\xi)}{A(n, Q)} \qquad (5)$$

where $A(n, y)=\int_0^\infty \{\beta(\omega)\}^n \exp(-\omega y) f(\omega) d\omega$

If we replace Q by ns on the right hand side of (4) and (5) we obtain a similar expression for $f(\omega \mid s,n)$ and $\varphi(\xi \mid s, n)$.

The p.d $\varphi(\xi \mid Q, n)$ possesses some very interesting stochastic order properties which are examined in theorem 2.1, 2.2 and 2.3. Karlin (1960), taking (Q, n) as a sufficient statistic for the parameter ω, stated and proved theorem 2.1a and 2.2. His proofs are based in a crucial way on the concept of "variation diminishing transformations". Here we give alternative proofs using the property of monotone likelihood ratio of the exponential family. Iglehart (1964), taking (s, n) as a sufficient statistic for the parameter ω, stated and proved theorem 2.1b. Here we give alternative proof to this result. Iglehart also stated a theorem parallel to 2.2 taking the pair (s, n) as a sufficient statistic for ω. His result is not valid, as it has shown in (Papachristos, 1977a). So taking (Q, n) as a sufficient statistic for ω we prove theorem 2.2. Theorem 2.3 is new and gives a (stochastic) upper bound for $\varphi(\xi| Q, n)$.

Theorem 2.1. *For any $n\geq 1$ and Q, Q', s, s' with $Q<Q'$ and $s<s'$ we have*

a) $\varphi(\xi/Q, n)\subset\varphi(\xi/Q', n)$

b) $\varphi(\xi/s, n) \subset\varphi(\xi/s', n)$

PROOF. If we set

$$\Delta(\xi, Q, Q')= \varphi(\xi| Q, n)- \varphi(\xi| Q', n)$$

then we must prove that

$$\int_0^x \Delta(\xi, Q, Q')d\xi \geq 0 \quad \forall x \in R.$$

Since $\varphi(\xi| Q, n)$ and $\varphi(\xi| Q', n)$ are p.d. we have

$$\int_0^\infty \Delta(\xi, Q, Q')d\xi = 0$$

This proves that the function $\Delta(\xi, Q, Q')$ changes sing at least once (from positive to negative or reversely) as ξ traverses the interval $[0, \infty)$. The function $\Delta(\xi, Q, Q')$ can be written

$$\Delta(\xi,Q, Q')= \int_0^\infty \varphi(\xi|\omega)\frac{\{\beta(\omega)\}^n \exp(-\omega Q')}{A(n,Q)}t_1(\omega)f(\omega)d\omega \tag{6}$$

with $t_1(\omega)=\exp\{-\omega(Q-Q')\}-\dfrac{A(n, Q)}{A(n, Q')}$

Since $Q<Q'$ it follows that $A(n, Q)>A(n, Q')$ and it is easy to see that if ω traverses the interval $[0, \infty)$ the function $t_1(\omega)$ changes sign exactly once from negative to positive values.

Now based on the two lemmas, given in the appendix we conclude that $\Delta(\xi, Q, Q')$ changes sign at most once from positive to negative values as ξ traverses the interval $[0, \infty)$. So $\Delta(\xi, Q, Q')$

changes sing exactly once from positive to negative values and this proves the theorem. Replacing (Q, Q′) with (ns, ns′) the part b of theorem follows

Theorem 2.2. *For any n≥1 and Q we have,*

$$\varphi(\xi/Q,\ n+1)\subset\varphi(\xi/Q,\ n)$$

PROOF. The proof is similar to that of the previous theorem, we only note the following

If

$$\Delta(\xi,\ n)=\varphi(\xi|\ Q,\ n+1)-\varphi(\xi|\ Q,\ n)$$

Then

$$\Delta(\xi,\ n)=\int_0^\infty \varphi(\xi|\omega)\frac{(\beta(\omega))^n\exp(-\omega Q)}{A(n+1,Q)}t_2(\omega)f(\omega)d\omega\,, \qquad (7)$$

with $t_2(\omega)=\beta(\omega)-\dfrac{A(n+1,Q)}{A(n,Q)}$. Since $\beta(\omega)$ is a strictly increasing function we have

$$\beta(0)A(n,\ Q)<A(n+1,\ Q).$$

But $\beta(\omega)\to\infty$ as $\omega\to\infty$, and so the function $t_2(\omega)$ changes sign once from negative to positive values as ω traverses the interval $[0,\ \infty)$.

Theorem 2.3. *For any n≥1 and Q, s we have*

a) $\varphi(\xi/Q,\ n)\subset\varphi(\xi/0)$

b) $\varphi(\xi/s,\ n)\subset\varphi(\xi/0)$

where $\varphi(\xi|\ 0)$ is given by (1) for $\omega=0$

PROOF. Let us set

$$\Delta(\xi)=\varphi(\xi|\ Q,\ n)-\varphi(\xi|\ 0)=r(\xi)\int_0^\infty(\beta(\omega)\exp(-\omega\xi)-\beta(0))f(\omega|Q,n)d\omega$$

Then we must prove that

$$\int_0^x \Delta(\xi)d\xi\ \geq 0\quad \forall\ x\in R.$$

But $\Delta(0)=r(0)\int_0^\infty(\beta(\omega)-\beta(0))f(\omega|Q,n)d\omega>0$ because $r(0)>0$ and $\beta(\omega)>\beta(0)$

Since

$$\int_0^\infty\Delta(\xi)d\xi=0$$

It follows that the function $\Delta(\xi)$ changes sign at least once as ξ traverses the interval $[0,\infty)$

Moreover the function $\beta(\omega)\exp(-\omega\xi)-\beta(0)$ is strictly decreasing with respect to ξ and goes to $-\beta(0)$ as $\xi\to\infty$. So $\Delta(\xi)$ changes sign exactly once from positive to negative values and this proves the theorem.

3. THE RANGE FAMILY OF TYPE A

Let ξ be a continuous random variable with density

$$g(\xi|\omega)=\beta(\omega)q(\xi)\psi(\omega,\xi),\ \xi\geq 0 \tag{8}$$

where

$$\psi(\omega,\xi)=\begin{cases}1 \text{ if } \xi\leq\omega\\ 0 \text{ if } \omega<\xi\end{cases},$$

$$(\beta(\omega))^{-1}=\int_0^\omega q(\xi)d\xi, \tag{9}$$

ω is an unknown parameter taking values in some interval $I_2=[0,\infty)$ and $q(\xi)$ is a strictly positive known function, such that the integral in (9) exists for every $\omega\in I_2$. We call this family the "the range family of type A". Defining suitably the function $q(\xi)$ we can obtain from this family some known upper truncated distributions.

Let $\xi_1, \xi_2,\ldots, \xi_n$ be the values of n independent observations of the random variable ξ whose density is $g(\xi|\omega)$. The likelihood function of $\xi_1, \xi_2,\ldots, \xi_n$ is

$$g(\xi_1, \xi_2,\ldots, \xi_n|\omega)=(\beta(\omega))^n\prod_{i=1}^n q(\xi_i)\psi(\omega,\upsilon)$$

where

$$\upsilon = \max_{1\le i\le n} \xi_i$$

This expression shows that the pair (υ, n) is a sufficient statistic for the parameter ω and also that υ is the maximum likelihood estimator (m.l.e.) based on $\xi_1, \xi_2, \ldots, \xi_n$ (Raiffa and Schlaifer,1961).

If $f(\omega)$ is the prior distribution for ω then using the Bayes theorem we find, the posterior density of ω

$$f(\omega \mid \upsilon, n) = \frac{\{\beta(\omega)\}^n f(\omega)\psi(\omega, \upsilon)}{\int_0^\infty \{\beta(\omega)\}^n f(\omega)\psi(\omega, \upsilon)d\omega}$$

$$= \begin{cases} \dfrac{\{\beta(\omega)\}^n f(\omega)}{B(\upsilon, n)} & \text{if } \omega \ge \upsilon \\ 0 & \text{if } \omega < \upsilon \end{cases}$$

where

$$B(\upsilon, n) = \int_\upsilon^\infty \{\beta(\omega)\}^n f(\omega)d\omega$$

The p.p.d.f. of ξ is now given by

$$g(\xi| \upsilon, n) = q(\xi)\frac{B(\max(\xi, \upsilon), n+1)}{B(\upsilon, n)} \quad \upsilon<\infty$$

This family was first studied by Karlin (1960) and later by Iglehart (1964) who stated and proved theorems 3.1 and 3.2 which follows. The proofs given by these authors are based on the concept of "variation diminishing transformation". Here we give alternatives proofs, which do not require this concept. Theorem 3.3 is new and gives an (stochastic) upper bound of p.p.d.f of the family (8).

Theorem 3.1. *For any $n\ge 1$ and υ, υ' with $\upsilon< \upsilon'<\infty$ we have*

$$g(\xi/\upsilon, n) \subset g(\xi/ \upsilon', n).$$

PROOF. We suppose that $0< \upsilon<\infty$, and let us set

$$\Delta(\xi, \upsilon, \upsilon') = g(\xi \mid \upsilon, n) - g(\xi \mid \upsilon', n).$$

Then we must prove that

$$\int_0^x \Delta(\xi, \upsilon, \upsilon')d\xi \geq 0 \;\forall\, x\,.$$

Obviously

$$\int_0^\infty \Delta(\xi, \upsilon, \upsilon')d\xi = 0$$

This last relation shows that $\Delta(\xi, \upsilon, \upsilon')$ change sing at least once as ξ traverses the interval $[0, \infty)$. The result will follow if we prove that the function $\Delta(\xi, \upsilon, \upsilon')$ changes sign exactly once from positive to negative values as ξ traverses the interval $[0, \infty)$.

The function $\Delta(\xi, \upsilon, \upsilon')$ can be written as

$$\Delta(\xi, \upsilon, \upsilon')=q(\xi)\, S(\xi, \upsilon, \upsilon')$$

where

$$S(\xi, \upsilon, \upsilon')=\frac{B(\max(\xi, \upsilon), n+1)}{B(\upsilon, n)}-\frac{B(\max(\xi, \upsilon'), n+1)}{B(\upsilon', n)}$$

It is obvious that $B(\upsilon, n)>B(\upsilon', n)$.

If $\xi<\upsilon$ then we have

$$S(\xi, \upsilon, \upsilon')=\frac{B(\upsilon, n+1)}{B(\upsilon, n)}-\frac{B(\upsilon', n+1)}{B(\upsilon', n)}$$

We shall prove that this is positive

If

$$D=B(\upsilon, n+1)B(\upsilon', n)-B(\upsilon, n)B(\upsilon', n+1)$$

then

$$D= B(\upsilon', n)\Big(B(\upsilon', n+1)+ \int_\upsilon^{\upsilon'}(\beta(\omega))^{n+1} f(\omega)d\omega\Big) - \Big(B(\upsilon', n)+ \int_\upsilon^{\upsilon'}(\beta(\omega))^{n} f(\omega)d\omega \Big)\, B(\upsilon', n+1)$$

$$= B(\upsilon', n) \int_\upsilon^{\upsilon'}(\beta(\omega))^{n+1} f(\omega)d\omega - B(\upsilon', n+1) \int_\upsilon^{\upsilon'}(\beta(\omega))^{n} f(\omega)d\omega$$

$$>\beta(\upsilon')\, B(\upsilon', n) \int_\upsilon^{\upsilon'}(\beta(\omega))^{n} f(\omega)d\omega - \beta(\upsilon')\, B(\upsilon', n) \int_\upsilon^{\upsilon'}(\beta(\omega))^{n} f(\omega)d\omega = 0$$

since $\beta(\omega)$ is strictly decreasing. So $S(\xi, \upsilon, \upsilon')>0$.

If $\upsilon \leq \xi \leq \upsilon'$ then

$$S(\xi, \upsilon, \upsilon') = \frac{B(\xi, n+1)}{B(\upsilon, n)} - \frac{B(\upsilon', n+1)}{B(\upsilon', n)}$$

is monotonically decreasing.

If $\upsilon' \leq \xi$ then

$$S(\xi, \upsilon, \upsilon') = \frac{B(\xi, n+1)}{B(\upsilon, n)} - \frac{B(\xi, n+1)}{B(\upsilon', n)} < 0$$

So the function $\Delta(\xi, \upsilon, \upsilon')$ changes sign once from positive to negative values as ξ traverses the interval $[0, \infty)$

The case $\upsilon=0$ can be treated in exactly the same way

Theorem 3.2. *For any* $n \geq 1$

a) $g(\xi/\upsilon, n+1) \subset g(\xi/\upsilon, n)$, *if* $\upsilon<\infty$

b) $g(\xi/\upsilon, n+1) = g(\xi/\upsilon, n)$, *if* $\upsilon=\infty$

PROOF. Suppose first that $0<\upsilon<\infty$. We follow the same line of proof as in theorem 3.1. So if

$$\Delta(\xi, \upsilon, n) = g(\xi \mid \upsilon, n+1) - g(\xi \mid \upsilon, n)$$

we are required to prove that

$$\int_0^x \Delta(\xi, \upsilon, n) d\xi \geq 0 \ \forall\ x.$$

The result will follow if we prove that the function $\Delta(\xi, \upsilon, n)$ changes sign exactly once from positive to negative values as ξ traverses the interval $[0, \infty)$. Again $\int_0^x \Delta(\xi, \upsilon, n) d\xi = 0$. The function $\Delta(\xi, \upsilon, n)$ can be written as

$$\Delta(\xi, \upsilon, n) = q(\xi)\, S(\xi, \upsilon, n)$$

where

$$S(\xi, \upsilon, n) = \frac{B(\max(\xi, \upsilon), n+2)}{B(\upsilon, n+1)} - \frac{B(\max(\xi, \upsilon), n+1)}{B(\upsilon, n)}$$

For $\xi \leq \upsilon$ we have

$$S(\xi, \upsilon, n)=\frac{B(\upsilon, n+2)}{B(\upsilon, n+1)}-\frac{B(\upsilon, n+1)}{B(\upsilon, n)}$$

This is positive because

$D= B(\upsilon, n+2)B(\upsilon, n)-[B(\upsilon, n+1)]^2$

$$= \int_{\upsilon}^{\infty}[(\beta(\omega))^{\frac{n+2}{2}}(f(\omega))^{\frac{1}{2}}]^2 d\omega \int_{\upsilon}^{\infty}[(\beta(\omega))^{\frac{n}{2}}(f(\omega))^{\frac{1}{2}}]^2 d\omega - [B(\upsilon, n+1)]^2$$

$$> \left[\int_{\upsilon}^{\infty}(\beta(\omega))^{n+1} f(\omega) d\omega\right]^2 -[B(\upsilon, n+1)]^2=0$$

where the last inequality was obtained using Schwartz inequality for integrals.

For $\xi \geq \upsilon$ we have

$$S(\xi, \upsilon, n)=\frac{B(\xi, n+2)}{B(\upsilon, n+1)}-\frac{B(\xi, n+1)}{B(\upsilon, n)}$$

and the function $S(\xi, \upsilon, n)$ is strictly decreasing. Since $\beta(\omega)$ is a decreasing function it follows that

$$S(\xi, \upsilon, n)< B(\xi, n+1)[\frac{\beta(\xi)}{B(\upsilon, n+1)}-\frac{1}{B(\upsilon, n)}]$$

and taking ξ large enough we can make $S(\xi, \upsilon, n)$ negative because $\beta(\infty)B(\upsilon, n)<B(\upsilon, n+1)$

Similar arguments can give the proof for the case $\upsilon=0$

If $\upsilon=\infty$ then since $\upsilon \leq \omega$ it follows that $\omega=\infty$ and so

$$g(\xi \mid \upsilon, n)=\beta(\infty)q(\xi)$$

independently of n which proves the relation.

Theorem 3.3. *For any $n \geq 1$ and υ,*

$g(\xi/\upsilon, n) \subset g(\xi/\infty)$, where $g(\xi/\infty)$ is given by (8) for $\omega=\infty$

PROOF. From (8) we have

$$g(\xi| \infty)=\beta(\infty)q(\xi),\ \xi \geq 0$$

if

$$\Delta(\xi)= g(\xi| \upsilon, n)- g(\xi| \infty)$$

Then

$$\Delta(\xi)=q(\xi)\ [\frac{B(\max(\xi,\upsilon),n+1)}{B(\upsilon,n)}-\beta(\infty)]$$

Suppose first that $0<\upsilon<\infty$. Since $\beta(\omega)$ is a decreasing function we shall have

$$B(\upsilon, n+1)>\beta(\infty)B(\upsilon, n)$$

So if $\xi<\upsilon$ then $\Delta(\xi)>0$. If $\xi>\upsilon$ then the function

$$A(\xi)=\ [\frac{B(\xi,n+1)}{B(\upsilon,n)}-\beta(\infty)]$$

is strictly decreasing in ξ and $\lim_{\xi\to\infty} A(\xi)=-\beta(\infty)$. So $\Delta(\xi)$ changes sign once from positive to negative values as ξ traverses the interval $[0, \infty)$.

If $\upsilon=0$ then

$$\Delta(\xi)=q(\xi)\ [\frac{B(\xi,n+1)}{B(0,n)}-\beta(\infty)]$$

It is easy to see that for sufficiently small values of ξ $\Delta(\xi)$ is positive. This and the fact that $B(\xi, n+1)$ decreases monotonically to zero as $\xi\to\infty$, while $\beta(\infty)>0$ establish the result.

4. THE RANGE FAMILY OF TYPE B

Let ξ be a continuous random variable with density belonging to the family

$$\sigma(\xi|\ \omega)=\gamma(\omega)q(\xi)\psi(\omega, \xi),\ \xi\geq 0 \tag{10}$$

where

$$\psi(\omega,\xi)=\begin{cases}1 \text{ if } \omega\leq\xi<\infty\\ 0 \text{ if } \xi<\omega\end{cases},$$

$$(\gamma(\omega))^{-1}=\int_{\omega}^{\infty}q(\xi)d\xi, \tag{11}$$

ω is an unknown parameter taking values in the interval $I_3=[0,\infty)$ and $q(\xi)$ is a strictly positive and bounded function on $[0,\infty)$, such that the integral in (11) exists for every $\omega \in I_3$. We call this family the "the range family of type B". Defining suitably the function $q(\xi)$ we can obtain from this family some known below truncated distributions.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be the values of n independent observations of the random variable ξ whose p.d. is $\sigma(\xi|\omega)$, and $\tau = \min_{1\le i\le n} \xi_i$. The likelihood function of $\xi_1, \xi_2, \ldots, \xi_n$ is

$$\sigma(\xi_1, \xi_2, \ldots, \xi_n | \omega) = (\gamma(\omega))^n \prod_{i=1}^{n} q(\xi_i)\psi(\omega, \tau)$$

This expression shows that the pair (τ, n) is a sufficient statistic for the parameter ω and also that τ is the m.l.e of ω based on $\xi_1, \xi_2, \ldots, \xi_n$.

If $f(\omega)$ is the prior p.d. of ω then using the Bayes theorem we find, for $\tau>0$, the posterior density of ω

$$f(\omega | \tau, n) = \frac{(\gamma(\omega))^n f(\omega)\psi(\omega, \tau)}{\int_0^\infty (\gamma(\omega))^n f(\omega)\psi(\omega, \tau) d\omega}$$

$$= \begin{cases} \dfrac{(\gamma(\omega))^n f(\omega)}{A(\tau, n)} & \text{if } \omega \le \tau \\ 0 \quad \text{if } \omega > \tau \end{cases}$$

where

$$A(\tau, n) = \int_0^\tau (\gamma(\omega))^n f(\omega)\, d\omega$$

The p.p.d.f. of ξ is now given by

$$\sigma(\xi | \tau, n) = q(\xi)\frac{A(\min(\xi, \tau), n+1)}{A(\tau, n)}, \quad \xi\ge 0, \quad \tau>0.$$

We give an example of a distribution belonging to this family and the respective p.p.d.f.

Example. If we define the density function $\sigma(\xi|\omega)=\exp(\omega-\xi)$ $\omega<\xi<\infty$ with a prior for parameter ω $f(\omega)=\exp(-\omega)$ then the p.p.d.f is given by

$$\sigma(\xi | \tau, n) = \begin{cases} \dfrac{n-1}{n}\dfrac{\exp[(n-1)\xi] - \exp(-\xi)}{\exp[(n-1)\tau]-1} & \xi < \tau \\[2ex] \dfrac{n-1}{n}\exp(-\xi)\dfrac{\exp(n\tau)-1}{\exp[(n-1)\tau]-1} & \xi > \tau \end{cases}$$

The study of this family was suggested by Iglehart (1964) in his pioneering article. In the next three theorems we establish results parallel to those obtained by Iglehart for the range family of type A.

Theorem 4.1. *For any $n \geq 1$ and τ, τ' with $\tau < \tau'$ we have,*

$$\sigma(\xi/\tau, n) \subset \sigma(\xi/\tau', n)$$

PROOF. If we set

$$\Delta(\xi, \tau, \tau') = \sigma(\xi|\tau, n) - \sigma(\xi|\tau', n)$$

Then we must prove that

$$\int_0^x \{\sigma(\xi|\tau, n) - \sigma(\xi|\tau', n)\} d\xi \geq 0 \ \forall \ x$$

The theorem will be valid if we prove that $\Delta(\xi, \tau, \tau')$ changes sign exactly once from positive to negative values as ξ traverse the interval $[0, \infty)$ (Again it is easily seen that $\int_0^\infty \Delta(\xi, \tau, \tau') d\xi = 0$)

Consider first the case $\tau > 0$. The difference $\Delta(\xi, \tau, \tau')$ can be written as

$$\Delta(\xi, \tau, \tau') = q(\xi) \, S(\xi, \tau, \tau')$$

where

$$S(\xi, \tau, \tau') = \frac{A(\min(\xi, \tau), n+1)}{A(\tau, n)} - \frac{A(\min(\xi, \tau'), n+1)}{A(\tau', n)}$$

If $0 \leq \xi \leq \tau$ then

$$S(\xi, \tau, \tau') = \frac{A(\xi, n+1)}{A(\tau, n)} - \frac{A(\xi, n+1)}{A(\tau', n)} > 0$$

since $A(\tau, n) < A(\tau', n)$

If $\tau \leq \xi \leq \tau'$ then

$$S(\xi, \tau, \tau') = \frac{A(\tau, n+1)}{A(\tau, n)} - \frac{A(\xi, n+1)}{A(\tau', n)}$$

and is strictly decreasing in ξ.

If $\tau' \leq \xi$ then

$$S(\xi, \tau, \tau')=\frac{A(\tau, n+1)}{A(\tau, n)}-\frac{A(\tau', n+1)}{A(\tau', n)}<0.$$

This last inequality can be established as follows.

If

$$D=A(\tau, n+1)A(\tau', n)-A(\tau, n)A(\tau', n+1)$$

then

$$D= A(\tau, n+1)\{ A(\tau, n)+ \int_{\tau}^{\tau'}(\gamma(\omega))^{n} f(\omega)d\omega) - \{ A(\tau, n+1)+ \int_{\tau}^{\tau'}(\gamma(\omega))^{n+1} f(\omega)d\omega\ ) A(\tau, n)$$

$$= A(\tau, n+1) \int_{\tau}^{\tau'}(\gamma(\omega))^{n} f(\omega)d\omega - A(\tau, n) \int_{\tau}^{\tau'}(\gamma(\omega))^{n+1} f(\omega)d\omega$$

$$<\gamma(\tau) A(\tau, n) \int_{\tau}^{\tau'}(\gamma(\omega))^{n} f(\omega)d\omega - \gamma(\tau) A(\tau, n) \int_{\tau}^{\tau'}(\gamma(\omega))^{n} f(\omega)d\omega =0$$

since $\gamma(\omega)$ is strictly increasing. So in the interval $[\tau, \tau']$ there will exist a unique point ξ_0 such that $S(\xi_0, \tau, \tau')=0$ and so $\Delta(\xi, \tau, \tau')\geq(\leq)0$ for $\xi\geq\xi_0$ ($\xi\leq\xi_0$)

If $\tau=0$, then since $0\leq\omega\leq\tau$ we have $\omega=0$ and

$$\sigma(\xi| 0, n)=\gamma(0)q(\xi), \quad \xi\geq 0$$

So

$$\Delta(\xi, \tau, \tau')=q(\xi)[\gamma(0)- \frac{A(\min(\xi, \tau'), n+1)}{A(\tau', n)}]$$

The quantity inside the brackets is a decreasing function of ξ, because $A(\tau, n)$ is an increasing function of τ and $\min(\xi, \tau')$ is increasing in τ'. Moreover for $\xi\geq\tau'$ it is negative, while for ξ close to zero it is positive. So it changes sign exactly once as ξ traverses the interval $[0, \infty)$ and this prove the theorem

Theorem 4.2. *For any $n\geq 1$ and τ we have*

a) $\sigma(\xi/\tau, n)\subset\sigma(\xi/\tau, n+1)$, if $\tau>0$

b) $\sigma(\xi/\tau, n)= \sigma(\xi/\tau, n+1)$, if $\tau=0$

PROOF. Let us set

$$\Delta(\xi, \tau, n)= \sigma(\xi| \tau, n)- \sigma(\xi| \tau, n+1)=q(\xi)S(\xi, \tau, n), \text{and } \tau>0$$

Where

$$S(\xi, \tau, n)= \frac{A(\min(\xi, \tau), n+1)}{A(\tau, n)} - \frac{A(\min(\xi, \tau), n+2)}{A(\tau, n+1)}$$

Similar with theorem 2.1 we are required to prove that

$$\int_0^x \Delta(\xi,\tau, n)d\xi > 0 \ \forall \ x.$$

If $\xi \leq \tau$ then

$$S(\xi, \tau, n)= \int_0^\xi (\gamma(\omega))^{n+1} f(\omega)[\frac{1}{A(\tau, n)} - \gamma(\omega)\frac{1}{A(\tau, n+1)}]d\omega$$

Since $\gamma(\omega)$ is strictly increasing $A(\tau, n+1) > \gamma(0)A(\tau, n)$ and so for $\xi \in [0, \varepsilon]$, where $\varepsilon > 0$ is a suitably small number, we have $S(\xi, \tau, n) > 0$.

If $\tau \leq \xi$ then

$$S(\xi, \tau, n)= \frac{A(\tau, n+1)}{A(\tau, n)} - \frac{A(\tau, n+2)}{A(\tau, n+1)} < 0$$

This follows because

$$A(\tau, n)A(\tau, n+2)= \int_0^\tau ((\gamma(\omega))^{n/2}(f(\omega))^{1/2})^2 d\omega \int_0^\tau ((\gamma(\omega))^{(n+2)/2}(f(\omega))^{1/2})^2 d\omega$$

$$> [\int_0^\tau (\gamma(\omega))^{n+1} f(\omega)d\omega]^2 = [A(\tau, n+1)]^2$$

This last inequality is deduced by applying Schwartz 's inequality for integrals. So $S(\xi, \tau, n)$ changes sign once from positive to negative values in the interval $[0, \tau]$ and this prove the first part

If $\tau=0$ then $\sigma(\xi| 0, n)=\gamma(0)q(\xi)$ for all n

Theorem 4.3. *For any $n \geq 1$ and τ we have $\sigma(\xi/\tau, n) \subset \sigma(\xi/ \xi_1)$ where ξ_1 is the first given observation and $\sigma(\xi/ \xi_1)$ is given by (10) for $\omega = \xi_1$*

PROOF. Suppose first that $\tau > 0$.

If $\xi < \xi_1$ then $\sigma(\xi| \xi_1)=0$ and so

$$\sigma(\xi| \tau, n) - \sigma(\xi| \xi_1) > 0$$

If $\xi \geq \xi_1$ then

$$\sigma(\xi|\tau, n) - \sigma(\xi|\xi_1) = q(\xi)\left(\frac{A(\tau, n+1)}{A(\tau, n)} - \gamma(\xi_1)\right) < 0$$

This is negative because since $\tau \leq \xi_1$ and $\gamma(\omega)$ is strictly increasing

$$A(\tau, n+1) < \gamma(\tau)A(\tau, n) < \gamma(\xi_1)A(\tau, n)$$

So there is a jump from positive to negative values at the point $\xi = \xi_1$ and this prove the theorem for $\tau > 0$

If $\tau = 0$ then

$$\sigma(\xi|0, n) - \sigma(\xi|\xi_1) = q(\xi)\{\gamma(0) - \gamma(\xi_1)\}\psi(\xi_1, \xi)$$

and the same is true.

5. CONCLUDING REMARKS

In this article we have studied stochastic ordering for p.p.d.f. for a random variable ξ, with p.d. belonging to, exponential, range of type A and range of type B families of distributions. Members of these families are designated by an unknown parameter ω, which has a known prior density. Moreover all families accept a sufficient statistic for the parameter ω, which is based on full past history of observations on ξ.

These families of densities can be used to describe the demand fluctuations for various inventory systems. The fact that they accept a sufficient statistics, say w, gives the flexibility to use a Bayesian approach to study the inventory systems of interest. Scarf (1959 and 1960), Karlin (1960), Iglehart (1964), Papachristos (1977a) have proved that the stochastic ordering on the predictive densities, related to the values of w, is transferred into inequalities on the optimal ordering levels for some classes of inventory systems. This high level qualitative result, describes the behaviour of optimal levels in relation to the statistical information for the parameter ω contained in the sufficient statistics. The range family of type B studied here can very well serve to describe the demand for product in certain inventory systems. This is a quite suitable model in cases where some lower bound for the demand can be supposed. The results produced here for the range family of type B, can be used to extend the results of Scarf, Karlin and Iglehart for systems with demand described by the range family of type B. Iglehart (1964) also produced an asymptotic expansion for the optimal inventory level in the case of systems with infinite horizon. We do believe that a similar expansion can be established for the optimal inventory level in the case of the range family of type B. The results contained in theorems 2.3, 3.3, 4.3 can be used to establish in a more rigorous mathematical way the adaptive dynamic programming equation, which is used in the study of inventory systems. They also could be used in finding bounds on, (i) the optimal inventory levels, (ii) the rate of variation (derivatives) of cost functions and (iii) on the cost functions themselves of the models under study, which would be independent from the available statistical information for the parameter ω contained in the sufficient statistics.

Additionally to the above we believe that the reduction of the dimension of the state space, proved by Scarf (1960), Azoury (1985), Lariviere and Porteus (1999) could also be established for the range family of type B.

The case of sufficient statistics based on censored data is also very interested. This arises in inventory systems operating with lost sales status.

APPENDIX

Lemma 1. *The exponential family as defined by 1 has an increasing likelihood ratio*

The proof of this lemma is trivial and omitted

Lemma 2. *If $p(x/\omega)$ has an increasing likelihood ratio and $t(\omega)$ changes sign at most once in its domain, then, for any distribution function $F(\omega)$, the function*

$$g(x)=\int p(x/\omega)\ t(\omega) d\,F(\omega)$$

changes sign at most once. Moreover if $t(\omega)$ changes sign in some direction as ω traverses its domain, $g(x)$ changes sign in the opposite direction.

This lemma is a slight modification of that given by Karlin and Rubin, 1956 (p. 276) and its proof can be derived using the same reasoning.

REFERENCES

Azoury, K. S. (1985), Bayes solution to dynamic inventory models under unknown demand distribution. *Manag. Science*, 31, 1150-1160.

Boland, P.J. , El – Neweihi, E., and Proschan, F. (1992), Stochastic order for redundancy allocations in series and parallel systems, *Advances in applied probability* 24, 161-171.

Braden, D. J. and Freimer M. (1991) Informational dynamics of censored observations. *Manag. Science*, 17, 1390-1404.

Brown G. F. and W. F. Rogers (1972) A Bayesian approach to demand estimation and inventory provisioning *Nav. Res. Log. Quart.* 19, 607-624.

Dunsmorre, I. R. and Aitchison, J. (1975) *Statistical Prediction Analysis* Cambridge University Press.

Iglehart, D. L. (1964) The dynamic inventory problem with unknown demand distributions *Manag. Science*, 10, 429-440.

Karlin, S. (1957) Polya type distributions II. *Ann. Math. Stat.*, 28, 281-308.

Karlin, S. (1960) Dynamic inventory policy with varying stochastic demands *Manag Science*, 6, 231-258.

Karlin S., Rubin, H. (1956) The theory of decision procedures for distributions with a monotone likelihood ratio *Ann. Math Stat* 27, 272-299.

Lariviere, M. A. and Porteus E. L. (1999), Stalking information: Bayesian inventory management with unobserved lost sales *Manag. Science* 45, 346-363.

Mann, H. B. and Withney D. R. (1947) On a test of whether one of two random variables is stochastically larger than the other *Ann. Math Stat* 18, 50-60.

Masse, P. (1962) *Optimal investment decisions* Prentice Hall, Englewood Cliffs, NJ.

Nahmias, S. (1994) Demand estimation in lost sales inventory systems. *Nav. Res. Log.* 41, 739-757.

Papachristos, S. (1977a) *Adaptive dynamic programming and inventory control*. Ph.d theses Faculty of Economics, University of Manchester England.

Papachristos, S. (1977b) A note on the dynamic inventory problem with unknown demand distribution. *Manag. Science.* 23, 1248-1251.

Raiffa, H. and Schlaifer, R. (1961). *Applied Statistical Decision Theory.* Harvard Business School.

Ross, S. (1983) *Introduction to stochastic dynamic programming*. Academic New York.

Savage, L. J. (1972) *The foundations of statistics* 2nd ed. Dover. New York

Scarf, H. (1959) Bayes solutions of the statistical inventory problem *Ann. Math. Stat.* 30, 490-508.

Scarf, H. (1960) Some remarks on Bayes solutions to the inventory problem. *Nav. Res. Log. Quart.* 7, 591-596.

Shaked, M. and J. G. Shanthikumar (1994) *Stochastic orders and their applications* Academic New York.

Stoyan, D. (1983) *Comparison methods for queues and other stochastic models*, D. J. Dalley ed Willey New York.